COMBINATORICA 3 (2) (1983) 207—218

ON THE PLANAR INTEGER TWO-FLOW PROBLEM

M. V. LOMONOSOV

Received 23 January 1981
Revised 26 May 1982

We consider the two-commodity flow problem and give a good characterization of the optimum flow if the augmented network (with both source-sink edges added) is planar. We show that max flow $\geqq$ min cut -1, and describe the structure of those networks for which equality holds.

1. Introduction

1.1. Let V be a finite set of vertices, $[V]$ denotes the set of unordered, and (V) — the set of ordered pairs of distinct vertices, called *edges* and *arcs*, respectively. Consider a *network* (V, c) where $c\colon [V]\to \mathbf{Z}_+$ is a *capacity function*, and let $G(c)=(V, E(c))$, where $E(c)=\{e\in[V];\ c(e)>0\}$. For nonempty disjoint subsets X, Y of V, let $[X, Y]$ denote the set of edges with one end in X and one in Y. For a proper $X\subset V$ put $\delta X=[X, V-X]$, $\delta^+X=X\times(V-X)$, $\delta^-X=(V-X)\times X$. If A is a set, f is a function on A and $B\subseteqq A$, we write $f(B)$ instead of $\sum\{f(a);\ a\in B\}$. When $B=[X, Y]\subseteqq[V]$, we write $c[X, Y]$ instead of $c([X, Y])$.

For nonempty disjoint subsets X, Y of V, put

$$\min c(X, Y; c) = \min\{c(\delta U);\ X\subseteqq U\subseteqq V-Y\},$$

and for $Z\subseteqq[V]$ put

$$\min c(Z;\ c) = \min\{c(\delta U);\ U\subseteqq V,\ Z\subseteqq\delta U\},$$

if at least one such U exists (i.e. if the graph (V, Z) is bipartite). A set δU such that $Z\subseteqq\delta U$ and $c(\delta U)=\min c(Z; c)$ is called a *minimal Z-cut* (for c).

1.2. A *flow* with the set of sources X and set of sinks Y (or shortly, an (X, Y)-flow), where $X, Y\subset V$, $X\cap Y=\emptyset$, is a function $f\colon (V)\to\mathbf{R}_+$ with $\operatorname{div}_f(v)=0$ for $v\in V-(X\cup Y)$, $\geqq 0$ for $v\in X$, and $\leqq 0$ for $v\in Y$, where

$$\operatorname{div}_f(v) = \Sigma\{f(v, z)-f(z, v);\ z\in V-\{v\}\}.$$

The quantity $\operatorname{div}_f(X)=-\operatorname{div}_f(Y)$ is called the *norm* of f and is denoted by $\|f\|$.

AMS subject classification (1980): 90 B 10, 90 C 10

Let $Z \subseteq [V]$. A *Z-flow in* (V, c) is a family $F = \{f_z;\ z \in Z\}$ of flows, each with one source and one sink, where z is the (unordered) pair [source, sink] for f_z, satisfying the joint capacity constraints

$$\sum \{f_z(x, y) + f_z(y, x);\quad z \in Z\} \leqq c[x, y] \quad \text{for all} \quad [x, y] \in [V].$$

Put $\|F\| = \sum \{\|f_z\|;\ z \in Z\}$. Obviously,

$$\|F\| \leqq \min c(Z; c), \tag{1.2.1}$$

whenever the right hand side is defined.

1.3. Let $\max f^{(i)}(Z; c)$ denote the maximum of $\|F\|$ over all integer-valued Z-flows in (V, c), and let $\max f^{(r)}(Z; c)$ denote the same for real-valued Z-flows. Surely, $\max f^{(i)}(Z; c) \leqq \max f^{(r)}(Z; c)$.

In this paper $\max f^{(i)}(Z; c)$ is found for the special case when $|Z| = 2$ and the graph $G(c) \cup Z$ is planar. We shall prove that

$$\max f^{(i)}(Z; c) \geqq \min c(Z; c) - 1, \tag{1.3.1}$$

and equality holds only for a very special class of networks which is completely characterized.

Recall that in real numbers the well-known result of T. Hu [5] asserts that, if $|Z| = 2$, then, for arbitrary c,

$$\max f^{(r)}(Z; c) = \min c(Z; c),$$

and the simple counterexample in Fig. 1 shows that in integers this equality is not true. The graph Γ in Fig. 1 is K_4 subdivided by two additional vertices, s_1 and s_2, placed on non-adjacent edges of K_4; $Z = \{z_1, z_2\}$, where $z_i = [s_i, t_i]$, $i = 1, 2$, and $c_\Gamma(e) = 1$ for $e \in E(\Gamma) - Z$ and 0 otherwise. Surely Γ is planar, and we have min $c(Z; c_\Gamma) = 2$, while $\max f^{(i)}(Z; c) = 1$.

We shall also see that if $G(c) \cup Z$ is planar and $\max f^{(i)}(Z; c) < \min c(Z; c)$, then $G(c) \cup Z$ contains a subdivision H of Γ such that

$$\max f^{(i)}(Z; c - c_H) = \min c(Z; c - c_H) = \min c(Z; c) - 2,$$

where $c_H(e) = 1$ for $e \in E(H)$ and 0 otherwise.

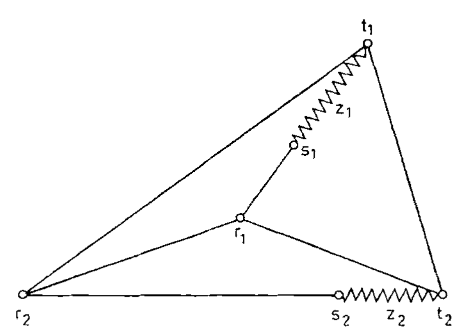

Fig. 1

1.4. Definition. (V, c) is called *Z-special* for a given $Z=\{z_1, z_2\}$ if there exists a 6-partition $(R_1, R_2, S_1, S_2, T_1, T_2)$ of V such that

(1.4.1) $z_\alpha \in [S_\alpha, T_\alpha], \quad \alpha = 1, 2;$

(1.4.2) $\delta(S_1 \cup S_2), \quad \delta(T_1 \cup T_2)$ and $\delta(S_1 \cup R_1 \cup T_2)$ are minimal Z-cuts of (V, c); and

(1.4.3) both $c(\delta R_1), c(\delta R_2)$ are odd.

Exemple: the network in Fig. 1 is Z-special.

1.5. Theorem 1. *Let (V, c) be a network and $Z=\{z_1, z_2\} \subset [V]$. Suppose that $G(c) \cup Z$ is planar. Then either*

$$\max f^{(i)}(Z; c) = \min c(Z; c), \quad or \tag{1.5.1}$$

$$\max f^{(i)}(Z; c) = \min c(Z; c) - 1; \tag{1.5.2}$$

moreover, (1.5.2) holds if and only if (V, c) is Z-special.

Theorem 2. *Let (V, c) be Z-special and $G(c) \cup Z$ be planar. Then there exists a subgraph H of $G(c) \cup Z$ with $Z \subset H$ such that*

(a) *H is a subdivision of K_4 with z_1, z_2 lying on nonadjacent (subdivided) edges of K_4 (shortly, H is "a subdivision of Γ")*
(b) $\min c(Z; c - c_H) = \min c(Z; c) - 2;$ *and*
(c) $(V, c - c_H)$ *is not Z-special.*

The first theorem clearly shows that (1.5.1) should be regarded as the "rule" and (1.5.2) as an "exception". This is emphasized by the following two sufficient conditions guaranteeing (1.5.1). The first of them is straightforward from Theorem 1.

Corollary 1. *Let $G(c) \cup Z$ be planar, where $Z=\{z_1, z_2\}$, $z_\alpha=[s_\alpha, t_\alpha]$, $\alpha=1, 2$. Put $A_s=\{s_1, s_2\}$, $A_t=\{t_1, t_2\}$, $B_1=\{s_1, t_2\}$, $B_2=\{s_2, t_1\}$. If*

$$\min c(A_s, A_t; c) \neq \min c(B_1, B_2; c), \tag{1.5.3}$$

then (1.5.1) holds.

The second sufficient condition will be derived from the proof of the main result.

Corollary 2. *Let $G(c) \cup Z$ be planar, where $Z=\{z_1, z_2\}$. If*

$$\min c(z_1; c) + \min c(z_2; c) \neq \min c(Z; c), \tag{1.5.4}$$

then (1.5.1) holds.

Theorems 1 and 2 may be applied to the following existence problem: given $d_1, d_2 \in \mathbf{Z}_+$, does there exist an integer Z-flow $F=\{f_1, f_2\}$ (where f_α stands for f_{z_α}) with $\|f_\alpha\|=d_\alpha$, $\alpha=1, 2$? Let $z_\alpha=[s_\alpha, t_\alpha]$, $\alpha=1, 2$. Add two new vertices s_1', s_2' to V and define c': $[V \cup \{s_1', s_2'\}] \to \mathbf{Z}_+$ by: $c'[s_\alpha', s_\alpha]=d_\alpha$, $c'[s_\alpha', v]=0$, $v \in V - \{s_\alpha\}$ for $\alpha=1, 2$; $c'(e)=c(e)$ for $e \in [V]$ and $c'[s_1', s_2']=0$. Put $Z'=\{[s_\alpha', t_\alpha]; \alpha=1, 2\}$.

Corollary 3. *Let* $G(c)\cup Z$ *be planar, and* $d_1, d_2\in\mathbf{Z}_+$. *An integer Z-flow* $F=\{f_1, f_2\}$ *with* $\|f_i\|=d_i$, $i=1, 2$, *exists if and only if*

$$\min c(Z; c') \geqq d_1+d_2, \qquad (1.5.5)$$

and the network $(V\cup\{s_1', s_2'\}, c')$ *is not Z-special. If (1.5.5) holds, but this network is Z-special then there exist integer Z-flows* $F'=\{f_1', f_2'\}$ *with* $\|f_1'\|=d_1-1$, $\|f_2'\|=d_2$ *and* $F''=\{f_1'', f_2''\}$ *with* $\|f_1''\|=d_1$, $\|f_2''\|=d_2-1$.

1.6. The question considered in this paper may be regarded as a special case of a general problem sounding as follows: to what extent $\max f^{(i)}(Z; c)$ depends on cuts of (V, c) cutting the edges of Z ? In real numbers, the class of Z's is characterized in [6] such that for every (V, c) $\max f^{(r)}(Z; c)$ is expressible, in the spirit of the linear programming duality, as a positive linear form of the quantities $\min c(X, Y; c)$, where $X, Y\subset V$ are covered by Z.

In integers, the following results are known for the case $|Z|=2$ and arbitrary networks:

(1.6.1) Existence of an integer Z-flow $F=\{f_1, f_2\}$ with prescribed $\|f_1\|$ and $\|f_2\|$ is NP-complete [3]. By the arguments used in 1.5, a good characterization of $\max f^{(i)}(Z; c)$ for arbitrary networks would imply the same for the integer two-flow existence problem.

(1.6.2) (Cherkasski [2]) Let $|Z|=2$ and $c(\delta\{v\})$ be even for each $v\in V$ not incident to Z. Then (1.5.1) holds.

(1.6.3) (P. D. Seymour [7]) Let $G=(V, E)$ be a graph, and let $Z\subseteq E$, $|Z|=2$. The equality (1.5.1) holds for every $c\colon [V]\to\mathbf{Z}_+$ satisfying $c(e)=0$ for all $e\in[V]-E$, if and only if G contains no subdivision of Γ.

Sections 2,3 contain simultaneous proofs of both Theorems and Corollary 2.

2. Preliminary considerations (not using planarity)

2.1. Proposition. *If* (V, c) *is Z-special then* $\max f^{(i)}(Z; c)<\min c(Z; c)$. *(Here* $G(c)\cup Z$ *is not necessarily planar.)*

Proof. Let (V, c) be Z-special and F be an integral Z-flow in (V, c) with the maximal $\|F\|$. Suppose that $\|F\|=\min c(Z; c)$. Define $a\colon [V]\to\mathbf{Z}_+$ by $a[x, y]=\sum_{\alpha=1,2}(f_\alpha(x, y)+f_\alpha(y, x))$. Then $a\leqq c$. Since F is integral $a(\delta X)$ is even for every $X\subset V$ with $(\delta X)\cap Z=\emptyset$. Further, for every $X\subset V$ with $Z\subseteq\delta X$ such that $c(\delta X)= =\min c(Z; c)$ we have $a(e)=c(e)$, $e\in\delta X$. This, in particular, is true for $X=S_1\cup S_2$, $T_1\cup T_2$ and $S_1\cup R_1\cup T_2$. Thus we have $a(e)=c(e)$ for $e\in[R_2, S_2]$, for $e\in[R_2, S_1\cup R_1\cup T_2]$ and for $e\in[R_2, T_1]$, since these sets are contained by $\delta(S_1\cup S_2)$, $\delta(S_1\cup R_1\cup T_2)$ and $\delta(T_1\cup T_2)$, respectively.

But $\delta R_2=[R_2, S_2]\cup[R_2, S_1\cup R_1\cup T_2]\cup[R_2, T_1]$. Hence $a(\delta R_2)=c(\delta R_2)$ which is impossible since the first is even and the second odd. Thus $\max f^{(i)}(Z; c)= \|F\|<\min c(Z; c)$, as required. ∎

2.2. The following notations wil be used. A *chain* $K=[v_0, v_1, \dots, v_k]$ is a graph with $V(K)=\{v_0, \dots, v_k\}$ and $E(K)=\{[v_{i-1}, v_i];\ i=1, \dots, k\}$. For distinct vertices x, y of K let $[xKy]$ denote the chain in K connecting x and y, and let $[xKy]$

and $]xKy[$ denote the "half-open" and "open" chains in K with the same edge-set as $[xKy]$ but without y or both x, y, respectively.

For two chains $[xKv]$ and $[yLv]$ with the unique common vertex v their concatenation is denoted by $[xKvLy]$;

The statement "$x, y, \ldots, z$ lie on K in this order" will be abbreviated as $(xy\ldots z)K$.

A *path* is a chain with a fixed order of its end-vertices; ordering the ends of K as (v_0, v_k) we obtain the path $(v_0, v_1, \ldots, v_k)$.

A path $P=(v_0, v_1, \ldots, v_p)$ is called a *line* of an (X, Y)-flow f if $v_0 \in X$, $v_p \in Y$ and $f(v_{i-1}, v_i) > 0$ for $i=1, \ldots, p$. Let $\mathcal{P}$ denote the set of all (X, Y)-paths (i.e. paths from X to Y). A *path-expansion* of f is a map $\varphi\colon \mathcal{P} \to \mathbf{R}_+$ such that $\varphi(P)>0$ only when P is a line of f, and the function f_0 defined by

$$f_0(x, y) = f(x, y) - \sum \{\varphi(P);\ P \in \mathcal{P}, (x, y) \ \text{ is an arc of } P\},$$

$(x, y) \in (V)$, is an (X, Y)-flow with the zero norm (i.e. is a flow without actual sources and sinks). Thus $\sum\{\varphi(P);\ P \in \mathcal{P}\} = \|f\|$. Given a path-expansion φ of f, a path P is called a line of φ if $\varphi(P)>0$ (in general not every line of f is a line of φ).

Let f be an integer (X, Y)-flow and P be an (X, Y)-path. "To *add P to f*" means to construct the (X, Y)-flow f' with

$$f'(x, y) = \begin{cases} f(x, y)+1, & \text{if } \ (x, y) \ \text{ is an arc of } P \text{ and } \ f(y, x) = 0, \\ f(x, y)-1, & \text{if } \ (y, x) \ \text{ is an arc of } P \text{ and } \ f(x, y) > 0, \\ f(x, y) & \text{otherwise.} \end{cases}$$

Given a line L of f, "to *remove L from f*" means to construct an (X, Y)-flow f' with

$$f'(x, y) = \begin{cases} f(x, y)-1, & \text{if } \ (x, y) \ \text{ is an arc of } L, \\ f(x, y) & \text{otherwise.} \end{cases}$$

Note that if f is obtained by adding P to some f' then P not necessarily can be removed from f.

2.3. Let us throughout assume that

(2.3.1) $$\max f^{(i)}(Z; c) < \min c(Z; c).$$

We need then to show that, first, there exists an integer Z-flow F with $\|F\| = \min c(Z; c) - 1$ and, second, that (V, c) is Z-special.

Let $z_\alpha = [s_\alpha, t_\alpha]$, $\alpha = 1, 2$. From (2.3.1) it is immediately seen that

(2.3.2) $$s_1, t_1, s_2, t_2 \ \text{ are all distinct.}$$

For if, say, $s_1 = s_2$, then any integral maximal flow f with source s_1 and sinks t_1, t_2 can be decomposed (i.e. by sorting the lines of some its path-expansion) into a sum $f_1 + f_2$, where f_α is a z_α-flow, $\alpha = 1, 2$, and we have $\|f_1\| + \|f_2\| = \|f\| = \min c(Z; c)$, by the max-flow min-cut theorem, contradicting (2.3.1).

(2.3.3) We assume, without loss of generality, that $c(z_1) = c(z_2) = 0$, so that $z_1, z_2 \notin G(c)$.

Indeed, a maximal Z-flow $\{f_1, f_2\}$ can always be chosen so that, assuming that, say, s_α is the source and t_α is the sink of f_α, $\alpha=1, 2$, we would have $f_\alpha(s_\alpha, t_\alpha)=c(z_\alpha)$; on the other hand, z_1, z_2 belong to every Z-cut.

2.4. Put $A_s = \{s_1, s_2\}$, $A_t = \{t_1, t_2\}$, $B_1 = \{s_1, t_2\}$, $B_2 = \{s_2, t_1\}$.

Let f be an integer maximal (B_1, B_2)-flow in (V, c) so that

(2.4.1) $$\|f\| = \min c(B_1, B_2; c) \geqq \min c(Z; c),$$

by the max-flow min-cut theorem [4] and since $Z \subset [B_1, B_2]$.

Consider a decomposition

(2.4.2) $$f = f_1 + f_2 + f_s + f_t$$

of f into one source, one sink flows where f_1, f_s have the source s_1; f_2, f_t have the source t_2; f_1, f_t have the sink t_1; and f_2, f_s have the sink s_2, with

(2.4.3) $$\|f\| = \|f_1\| + \|f_2\| + \|f_s\| + \|f_t\|.$$

(This can be done by sorting the lines of some path-expansion of f.)

Put $F = \{f_1, f_2\}$. We have

(2.4.4) $$\|f_s\| + \|f_t\| > 0,$$

by (2.3.1) and (2.4.1).

(2.4.5) We assume, without loss of generality, that $\|f_t\| \geqq \|f_s\|$, whence $\|f_t\| > 0$.

2.5. Let f and its decomposition (2.4.2) be chosen so that

(2.5.1) $\|f\|$ is as small as possible under (2.4.5), and

(2.5.2) $\|f_s\|$ is as large as possible under (2.4.5) and (2.5.1).

2.6. Let V_s denote the union of A_s and the set of vertices lying on the lines of f_s (if f_s is non-zero), and let V_t be defined similarly. For $x, y \in V_s$ (V_t), let $x \leqq y$ mean that one of the following situations takes place:

(a) $x = s_1$ (resp., t_1),

(b) $y = s_2$ (resp., t_2),

(c) $x = y$,

(d) there exists a line L of f_s (resp., f_t) such that $(s_1 x y s_2) L$ (resp., $(t_1 x y t_2) L$).

(Thus in general $x \leqq y$ and $y \leqq x$ does not imply $x = y$; however, if $x \neq y$ then x, y lie on a circuit of f_s resp., f_t).)

As an immediate consequence of (2.5.1) we obtain

(2.6.1) **Proposition.** $V_s \cap V_t = \emptyset$.

Proof. Suppose not, and let $v \in V_s \cap V_t$. If there exist a line J of f_t and a line K of f_s both incident to v, then $J \cup K$ contains two edge disjoint chains: L_α with the ends s_α, t_α, $\alpha = 1, 2$. Remove J from f_t and K from f_s; add L_α to f_α, $\alpha = 1, 2$. This gives another decomposition f_1', f_2', f_s', f_t' with $\|f_t'\| = \|f_t\| - 1$, $\|f_s'\| = \|f_s\| - 1$, contradicting (2.5.1).

If v does not belong to a line of some of f_s, f_t, then this must be f_s, by (2.4.5), and then f_s is zero and $v \in A_s$. Let $v = s_1$, say, and let J be a line of f_t incident to v.

Remove J from f_t and add $[s_1 J t_1]$ to f_1. This will diminish $\|f_t\|$ preserving (2.4.5) contradicting (2.5.1). ∎

2.7. An (V_s, V_t)-path $P=(v_0, v_1, \ldots, v_p)$ is called an *F-path* if $\{v_1, \ldots, v_{p-1}\} \subseteqq V - V_s - V_t$ and each arc of P is assigned to one of the three classes: *forward, backward*[1] and *backward*[2] so that

$$c[x, y] - f_1(x, y) - f_2(y, x) > 0 \quad \text{if} \quad (x, y) \quad \text{is forward,}$$
$$f_1(y, x) > 0 \quad \text{if} \quad (x, y) \quad \text{is backward}^1\text{, and}$$
$$f_2(x, y) > 0 \quad \text{if} \quad (x, y) \quad \text{is backward}^2.$$

(Recall that $F=\{f_1, f_2\}$ where f_1 is directed from s_1 to t_1, while f_2 is directed from t_2 to s_2.)

A family $\mathscr{L}$ of not necessarily distinct F-paths is *independent* if for each $(x, y) \in (V)$ the number of paths from $\mathscr{L}$ containing (x, y) as a

$$\left.\begin{matrix}\text{forward}\\ \text{backward}^1\\ \text{backward}^2\end{matrix}\right\} \text{ arc does not exceed } \begin{cases} c[x, y] - f_1(x, y) - f_2(y, x) \\ f_1(y, x) \\ f_2(x, y). \end{cases}$$

(2.7.1) Define $c_f \colon (V) \to \mathbf{Z}_+$ by the formula

$$c_f(x, y) = c[x, y] - f_1(x, y) - f_2(y, x) + f_1(y, x) + f_2(x, y),$$

and consider the *directed network* (V, c_f). For non-empty $X, Y \subset V$ with $X \cap Y = \emptyset$ put

$$\overrightarrow{\min}\, c(X, Y; c_f) = \min \{c_f(\delta^+ Z) ; X \subseteqq Z \subseteqq V - Y\}.$$

(2.7.2) **Proposition.**

$$\overrightarrow{\min}\, c(V_s, V_t; c_f) \geqq \min c(Z; c) - \|F\|,$$

and there exist as many as $\overrightarrow{\min}\, c(V_s, V_s; c_f)$ *independent F-paths.*

Proof. Define an (s_2, t_2)-flow $\bar{f}_2$ by the relation $\bar{f}_2(x, y) = f_2(y, x)$, $(x, y) \in (V)$, and consider $f_1 + \bar{f}_2$ as a (V_s, V_t)-flow in (V, c). The following relation is almost obvious (its proof is in [1, Sect. 1.2]).

$$\min c(V_s, V_t; c) = \|f_1 + f_2\| + \overrightarrow{\min}\, c(V_s, V_t; c_f).$$

Since $A_s \subseteqq V_s$ and $A_t \subseteqq V_t$ we have $\min c(V_s, V_t; c) \geqq \min c(A_s, A_t; c) \geqq \min c(Z; c)$. Further, $\|f_1 + \bar{f}_2\| = \|f_1\| + \|\bar{f}_2\| = \|F\|$. This proves the inequality in (2.7.2).

Let h be an integer maximal (V_s, V_t)-flow in (V, c_f), so that $\|h\| = \overrightarrow{\min}\, c(V_s, V_t; c_f)$, by the max-flow min-cut theorem [4]. By the inequality just proven and (2.3.1), $\|h\| > 0$. Choose a path-expansion φ of h and let $\mathscr{L} = (P_i; i = 1, \ldots, \|h\|)$ be a list of lines of φ in which each line P appears precisely $\varphi(P)$ times. For arbitrary arc (x, y) let $(P_{i_j}; j = 1, \ldots, h(x, y))$ be the sublist of $\mathscr{L}$ enumerating the members of $\mathscr{L}$ passing through (x, y). Then consider (x, y) as backward[1] in

$(P_{i_j};\ 1 \leqq j \leqq f_1(y, x))$, as backward[2] in $(P_{i_j};\ f_1(y, x) < j < f_1(y, x) + f_2(x, y))$ and as forward in $(P_{i_j}; f_1(y, x) + f_2(x, y) < j \leqq h(x, y))$. Thus $\mathscr{L}$ becomes an independent family of as many as $\|h\| = \overrightarrow{\min}\ c(V_s, V_t; c_f)$ F-paths, as required. ▌

(2.7.3) **Proposition.** *Suppose that* $\|f_t\| > \|f_s\|$. *If an F-path with no backward*[2] *arcs starts from* $v_0 \in V_s$ *then* $v_0 \neq s_1$. *Similarly, with 1 and 2 exchanged.*

Proof. Let $P = (v_0, v_1, \ldots, v_p)$ be an F-path with $v_0 = s_1$ whose arcs are only forward and backward[1]. Then $v_p \neq t_1$, for otherwise P would be an augmenting path [4] for f, contradicting the maximality of f. Let J be a line of f_t incident to v_p. Remove J from f_t and add $[s_1 P v_p J t_1]$ to f_1. This will diminish $\|f_t\|$ preserving $\|f\|$ and (2.4.5), contradicting (2.5.1). ▌

(2.7.4) **Proposition.** $\|f_s\| \neq 0$.

Proof. By (2.7.2) and (2.3.1) there must exist an F-path, say $P = (v_0, v_1, \ldots, v_p)$. If $f_s = 0$ then $v_0 \in A_s$, say $v_0 = s_1$. Then, by (2.7.3), P has backward[2] arcs. Let v_i be the first vertex along P incident to a line of f_2, and let J be a line of f_2 passing through v_i. Remove J from f_2 and add $[s_1 P v_i J s_2]$ to f_s. This increases $\|f_s\|$ preserving $\|f\|$ and (2.4.5), contradicting (2.5.2). ▌

(2.7.5) Let $P = (x_0, \ldots, x_p)$ and $Q = (y_0, \ldots, y_q)$ be F-paths (distinct or not). Let us write $P \lesseqq Q$ to express that

(a) $x_0 \leqq y_0$ (in V_s) and $x_p \leqq y_q$ (in V_t); and
(b) P has no backward[2] arc and Q has no backward[1] arc.

In particular, $P \lesseqq P$ means that each arc of P is forward.

(2.7.6) **Proposition.** *No independent pair P, Q of F-paths satisfies* $P \lesseqq Q$. *In particular, if* $P \lesseqq P$ *then* $\min\{c[x, y] - f_1(x, y) - f_2(y, x) \mid (x, y)$ *is an arc of* $P\} = 1$.

Proof. Let $P = (x_0, \ldots, x_p)$, $Q = (y_0, \ldots, y_q)$ be independent F-paths satisfying $P \lesseqq Q$. Then, by (2.7.4) and (2.4.5), there exist a line J of f_t and a line K of f_s such that $(t_1 x_p y_q t_2) J$ and $(s_1 x_0 y_0 s_2) K$. Remove J from f_t and K from f_s, and add $[s_1 K x_0 P x_p J t_1]$ to f_1 and $[t_2 J y_q Q y_0 K s_2]$ to f_2. This will diminish $\|f_t\|$ preserving $\|f\|$ and (2.4.5), contradicting (2.5.1). ▌

3. Planarity arguments

3.1. Planarity imposes further crucial restrictions on the layout of F-paths. From this moment the graph $G(c) \cup Z$ is considered as drawn on a 2-sphere $\mathscr{S}$. Let us regard $\mathscr{S}$ as *slitted* along the two open curves representing z_1 and z_2. An open domain Ω of $\mathscr{S}$ will be called *simple* if both slits are in $\mathscr{S} - \Omega$. Given a chain L of $G(c)$ with the ends $s_\alpha, t_\alpha, \alpha = 1$ or 2, let $\Omega(L)$ denote the simple domain with the boundary $\partial\Omega(L) = \bigcup \{z\}$.

(3.1.1) A set $\mathscr{L}$ of (s_α, t_α)-paths of $G(c)$, $\alpha = 1$ or 2, is called *laminar* if for two distinct members L', L'' of $\mathscr{L}$ either $\Omega(L')$, $\Omega(L'')$ are disjoint or one of these domains contains the other.

Let φ be a non-negative function on the set of all (s_α, t_α)-paths of $G(c)$, $\alpha=1$ or 2. For an arbitrary chain L with the ends s_α, t_α define $\varphi^*(L)=\sum\{\varphi(P);\ P$ lies in the closure of $\Omega(L)\}$.

Let φ be called *laminar* if $\{P;\ \varphi(P)>0\}$ is laminar.

Let P_1, P_2 be two lines of φ such that $\Omega(P_1)\cap\Omega(P_2)\neq\emptyset$ and $\Omega(P_1)\cup\Omega(P_2)\neq\Omega(P_i)$, $i=1,2$. Put $Q_1=\partial(\Omega(P_1)\cap\Omega(P_2))-\{z_\alpha\}$ and $Q_2=\partial(\Omega(P_1)\cup\Omega(P_2))-\{z_\alpha\}$ and consider Q_1, Q_2 as (s_α, t_α)-paths. Define another function ψ by assigning $\psi(L)=\varphi(L)$ for $L\neq P_1, P_2, Q_1, Q_2$; $\psi(P_i)=\varphi(P_i)-1$ and $\psi(Q_i)=\varphi(Q_i)+1$, $i=1,2$. Then $\psi^*(L)\geqq\varphi^*(L)$ for all L and $\psi^*(Q_1)>\varphi^*(Q_1)$.

(3.1.2) **Proposition.** *f_α has a laminar path-expansion* ($\alpha=1,2$).

Proof. Let φ be a path-expansion of f_α with a maximal φ^* in the sense that by choosing another path-expansion of f_α no value of φ^* can be increased without decreasing some other. Then φ is laminar, for if P_1, P_2 are as in the previous paragraph then Q_1, Q_2 defined there are also lines of f_α so that ψ is another path-expansion of f_α. But this contradicts maximality of φ^*. ▮

3.2. In this sequel the flows f_1, f_2 are regarded as defined by laminar integral non-negative functions φ_1, φ_2 (on the sets of (s_1, t_1)- and (s_2, t_2)-paths respectively).

We require that these φ_1 and φ_2 satisfy the following additional condition: (3.2.1) the corresponding functions φ_1^* and φ_2^* are maximal provided that f_1 and f_2 satisfy (2.4.3), (2.5.1) and (2.5.2).

(3.2.2) **Proposition.** *Let L be a line of φ_1 (or φ_2). Then $\Omega(L)$ meets no line of f_2 (resp., f_1), f_s and f_t.*

Proof. Suppose that $\Omega(L)$ meets J where L is a line of φ_1 and J is a line of some $f'\in\{f_2, f_s, f_t\}$. Choose L so that $\Omega(L)$ is minimal with this property, i.e. J does not meet $\Omega(L')$ for whatever line L' of φ_1 satisfying $L'\prec L$. Let x, y be two common vertices of L and J such that $]xJy[\subset\Omega(L)$ and put $L'=(s_1LxJyLt_1)$. Remove L from f_1 and add L' to the resulting flow; remove J from f' and add $(s'JxLyJt')$ to the resulting flow where s', t' are the source and sink of f'. This increases $v(L')$ decreasing no other value of v, contradicting (3.2.1). ▮

(3.2.3) **Proposition.** *Let L be a line of φ_α, $\alpha=1$ or 2, and $P=(v_0, v_1, \dots, v_p)$ be an F-path. Suppose that L and P have common vertices v_j, v_k, $0\leqq j<k\leqq p$, such that $(sv_jv_kt)L$. Then $]v_jPv_k[$ does not meet $\Omega(L)$.*

Proof. Suppose that $]v_jPv_k[\cap\Omega(L)\neq\emptyset$; then v_j, v_k can be chosen so that $]v_jPv_k[\subset\Omega(L)$. Let, say, $\alpha=1$. Then, by (3.2.2), $]v_jPv_k[$ contains no backward[2] arcs. Remove L from f_1 and add $L'=(s_1Lv_jPv_kLt_1)$ to the resulting flow. This increases $v(L')$ contradicting (3.2.1). ▮

(3.2.4) **Remark.** In the above proof only properties of the open segment $]v_jPv_k[$ of P are used, so that (3.2.3) remains true without the assuption $v_0\in V_s$, $v_p\in V_t$.

(3.2.5) **Proposition.** *Every F-path has only forward arcs.*

Proof. Let $P=(v_0, ..., v_p)$ be an F-path with backward arcs, the first of them along P being (v_j, v_{j+1}). Suppose, without loss of generality, that (v_j, v_{j+1}) is backward[1] so that $f_1(v_{j+1}, v_j)>0$ and there exists a line L of φ_1 passing through (v_{j+1}, v_j). Define a chain K with the ends s_1, v_j as follows: if $v_0=s_1$ then $K=[s_1Pv_j]$; if $v_0\neq s_1$ then choose a line J of f_s passing through v_0 and put $K=[s_1Jv_0Pv_j]$. Now let Ω be the open simple domain with $\partial\Omega=\{z_1\}\cup[s_1Kv_jLt_1]$. Since K is outside $\Omega(L)$, by (3.2.2) and (3.2.3), $]s_1Lv_j[$ lies in the closure of Ω. But v_{j+1} can lie on neither $[v_jLt_1]$ nor $[v_0Pv_j]$ since L and P are not self-intersecting. Nor can v_{j+1} lie on J since P have no inner point in V_s. Thus $v_{j+1}\in\Omega$. By similar reasons and by (3.2.3), $[v_{j+1}Pv_p]$ have no vertex in common with $\partial\Omega$. But $v_p\in V_t$ is outside Ω, a contradiction. ∎

3.3. This is coup de grâce of the proof.

(3.3.1) **Proposition.** $\overrightarrow{\min}\, c(V_s, V_t; c_f)\geqq 2$.

Proof. Suppose not. Then $\overrightarrow{\min}\, c(V_s, V_t; c_f)=1$, by (2.3.1) and (2.7.2). Then $\|F\|=\min c(Z; c)-1$. Let $P=(v_0, ..., v_p)$ be an F-path and let K be a line of f_s passing through v_0 and J be a line of f_t passing through v_p, by (2.7.4), (2.4.5). Add $(s_1Kv_0Pv_pJt_1)$ to f_1. This adds 1 to $\|F\|$ contradicting (2.3.1). ∎

(3.3.2) Due to planarity, and since f_s, f_t are non-zero (by (2.7.4), (2.4.5)) and $V_s\cap V_t=\emptyset$ (by (2.6.1)), neither f_s nor f_t has a pair of lines whose union separates z_1 from z_2 on $\mathcal{S}$. Let Δ_s, Δ_t denote the minimal simple closed domains of $\mathcal{S}$ containing the lines of f_s, f_t respectively (so that, say, if $\|f_s\|=1$ then Δ_s is the line of f_s and if $\|f_s\|\geqq 2$ then $\partial\Delta_s$ is the union of two independent lines of f_s). Then $\Delta_s\cup\Delta_t\cup Z$ partitions the remaining part of $\mathcal{S}$ into two disjoint simple open domains, say $\mathcal{A}$ and $\mathcal{B}$. By planarity, every open F-path entirely lies either in $\mathcal{A}$ or in $\mathcal{B}$.

(3.3.3) **Proposition.** *Neither $\mathcal{A}$ nor $\mathcal{B}$ contains two independent F-paths.*

Proof. Let $P=(x_0, ..., x_p)$ and $Q=(y_0, ..., y_q)$ be two independent F-paths, both in $\mathcal{A}$ (say). Then x_0 and y_0 belong to the same line of f_s contained by $\partial\Delta_s$, and x_p, y_q belong to the same line of f_t contained by $\partial\Delta_t$. Let $x_0\preceq y_0$. If $x_p\preceq y_q$ then $P\preceq Q$, by (3.2.5), contradicting (2.7.6). Thus $x_p\succ y_q$, moreover $x_0\prec y_0$, for if $x_0=y_0$, we might say that $y_0\preceq x_0$. Since both P and Q are in $\mathcal{A}$ they, by planarity, have a common inner vertex, say $x_j=y_k$. But then $P'=(x_0Px_jQy_q)$ and $Q'=(y_0Qy_kPx_p)$ are independent F-paths satisfying $P'\preceq Q'$ contradicting (2.7.6). ∎

(3.3.4) **Proposition.** $\|f_s\|=\|f_t\|=1$ *and* $\min c(V_s, V_t; c_f)=2$.

Proof. By (3.3.3), there cannot exist more than two independent F-paths. Together with (3.3.1) this implies $\min c(V_s, V_t; c_f)=2$, and, by (3.3.3), from any two independent F-paths one belongs to $\mathcal{A}$ and the other to $\mathcal{B}$. Since both f_s anf f_t are non-zero, we have

(3.3.5) $$\min c(B_1, B_2; c) = \|f\| \geqq \|F\|+2 = \min c(V_s, V_t; c) \geqq \min c(Z; c).$$

Suppose that $\|f_t\|\geqq 2$. Then $\partial\Delta_t$ is the union of two independent lines of f_t, say J' and J'', so that $J'\subset\partial\mathcal{A}$, $J''\subset\partial\mathcal{B}$. Let $P=(x_0, ..., x_p)$ and $Q=(y_0, ..., y_p)$ be two independent F-paths where $P\subset\mathcal{A}$, $Q\subset\mathcal{B}$, so that $x_p\in J'$, $y_q\in J''$. There are two edge-disjoint chains K', K'', each either a segment of a line of f_s or a single vertex, which connect x_0, y_0, respectively with distinct members of A_s. Without loss of generality let K' connect x_0 with s_1 and K'' connect y_0 with s_2. Add $(s_1K'x_0Px_pJ't_1)$ to f_1 and $(t_2J''y_qQy_0K''s_2)$ to f_2. This will give a Z-flow F' with $\|F'\|=\|F\|+2\geqq\min c(Z;c)$, contradicting (2.3.1).

Thus $\|f_t\|=1$, whence, by (2.4.5) and (2.7.4), also $\|f_s\|=1$. ∎

(3.3.6) Let now L_s, L_t denote the lines of f_s, f_t respectively, and let us fix two independent F-paths $P=(x_0, \dots, x_p)$, $Q=(y_0, \dots, y_q)$ with $x_0\prec y_0$ and $x_p\succ y_q$, by (2.7.6). We have $\{x_0, y_q\}\neq\{s_1, t_1\}$ and $\{y_0, x_p\}\neq\{s_2, t_2\}$ for otherwise the paths L_s, L_t, P and Q could be used to increase $\|f_1\|$ or $\|f_2\|$ by 2, contrary to (2.3.1). Therefore the subgraph $H=Z\cup L_s\cup L_t\cup P\cup Q$ of $G(c)\cup Z$ is a subdivision of Γ, see Figure 2. Let C denote the cycle of $H-Z$, i.e. $C=[c_0Px_pL_ty_qQy_0L_sx_0]$, and for $x\in\{s_1, s_2, t_1, t_2\}$ let H_x denote the half-open chain of $H-Z$ connecting $x\in H_x$ with C, without its end-vertex on C (if $x\in C$ then $H_x=\emptyset$). Then f_s and f_t lie in distinct closed "half-spheres" defined by C. Further, F is a Z-flow in $(V, c-c_H)$ whence $\max f^{(i)}(Z; c-c_H)\geqq\|F\|$ while, on the other hand, $\min c(Z; c-c_H)\leqq\min c(Z; c)-\min c(Z; c_H)=\min c(Z; c)-2\leqq\|F\|$, by (3.3.5). This, together with Proposition 2.1, proves Theorem 2. ∎ ∎

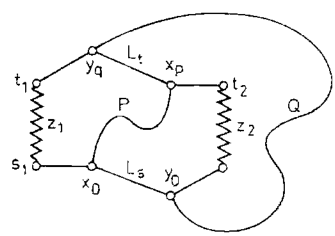

Fig. 2

(3.3.7) Now the above arguments and the resulting subgraph H are seen to be invariant under exchange of the roles of s_2 and t_2. (3.3.5) shows that there exists a minimal Z-cut separating V_s from V_t, say δU_1 with $V_s\subseteqq U_1$. By symmetry, there also exists a minimal Z-cut separating $[s_1L_sx_0Px_pL_tt_2]$ from $[s_2L_sy_0Qy_qL_tt_1]$, say δU_2 with $B_1\subseteqq U_2$.

(3.3.8) **Proposition.** *For* $\alpha=1, 2$, $\min c(z_\alpha, c)=\|f_\alpha\|+1$ and at least one of s_α, t_α is separated from C by a minimal z_α-cut.

Proof. Consider $\alpha=1$. Add $(s_1L_sx_0Px_pL_tt_1)$ to f_1 and let f_1' denote the flow thus obtained. We show that f_1' is a maximal z_1-flow in (V, c). Let a path $R=(z_0, z_1, \dots, z_r)$ be called *feasible* if for each $i=1, \dots, r$ either $f_1'(z_{i-1}, z_i)<c[z_{i-1}, z_i]$ or $f_1'(z_i, z_{i-1})>0$. Suppose that $z_0\in H_{s_1}$, $z_r\in H-H_{s_1}$ and $\{z_1, \dots, z_{r-1}\}$ does not meet H. Then R belongs to the component of $\mathcal{S}-C$ not containing f_2, whence $f_2(x, y)$, $f_2(y, x)$ are zero for each edge $[x, y]$ of R. Further, $z_r\in C\cup H_{t_1}$, by planarity.

We see that z_r cannot belong to $]y_0Qy_q[$ or $[t_1L_tx_p]$. For put $Q'=R$ if $z_r\in [t_1L_tx_p]$ and $Q'=(z_0Rz_rQy_q)$ if $z_r\in]y_0Qy_q[$. Then Q' and P are independent and $Q'\leqq P$ contradicting (2.7.6).

Thus $z_r \in]x_p P x_0 L_s y_0]$. Let s_1' denote the nearest to s_1 vertex of H_{s_1} from which $H-H_{s_1}$ can be achieved by a feasible path, and put $H_{s_1}'=[s_1 H_{s_1} s_1')$. Similarly, define H_{t_1}' in terms of feasible paths from $H-H_{t_1}$ to H_{t_1}. If both H_{s_1}', H_{t_1}' are empty, then let R_s and R_t be feasible paths from s_1 to $H-H_{s_1}$ and from $H-H_{t_1}$ to t_1, respectively. $H \cup R_s \cup R_t$ contains two independent F-paths from s_1 to t_1 contradicting (2.3.1). Thus, say, H_{s_1}' is non-empty.

Let S_1 denote the set of vertices $v \in V$ such that there exists a feasible path from H_{s_1} to v. Then δS_1 is a minimal z_1-cut in (V, c), for if $x \in S_1$, $y \in V-S_1$ and $c[x, y]>0$, then $f_1'(x, y)=c[x, y]$ and $f_1'(y, x)=0$ to prevent y from being achieved by a feasible path passing through (x, y).

Similar arguments work for s_2 and t_2.

(3.3.9) We may assume, due to the symmetry, that for each $\alpha=1, 2$ (V, c) has a minimal z_α-cut δS_α separating s_α from C. By the construction, $S_1, S_2 \subset U_1$, $S_1 \subset U_2$, $S_2 \subset V-U_2$ (cf. (3.3.7)). It is also seen that $\delta(S_1 \cup S_2)=\delta S_1 \cup \delta S_2$ is a minimal Z-cut of (V, c) and that $S_1 \cup S_2$ is a proper subset of U_1 since $V(L_s) \subseteqq U_1$. This proves Corollary 2.

Now, the atoms generated by U_1, U_2, S_1 and S_2 are:

$$S_1, S_2; \quad T_1 = V-U_1-U_2, \quad T_2 = U_2-U_1;$$

$$R_1 = U_1 \cap U_2 - S_1, \quad R_2 = U_1 - U_2 - S_1.$$

The c-capacities of the edges of δR_1 and δR_2 are saturated by f_1, f_2 and c_H, whence both $c(\delta R_1)$, $c(\delta R_2)$ are odd. This means that (V, c) is Z-special. This completes the proof of Theorem 1. ▮ ▮

Acknowledgement. The author is grateful to the referees for very helpful criticism and suggestions.

References

[1] G. M. Adelson-Velski, E. A. Dinitz and A. V. Karzanov, *Algorithms for Flows in Networks* (in Russian), Nauka, Moscow, 1975.

[2] B. V. Cherkasski, On Multiterminal Two-commodity Network Flow Problems, In: *Issledovaniya po Diskretnoi Optimizacii* (in Russian), Nauka, Moscow, 1976, 261—289.

[3] S. V. Even, A. Itai and A. Shamir, On the complexity of the timetable and multicommodity flow problem, *SIAM J. Comput.*, **5** (4) (1976).

[4] L. R. Ford and D. R. Fulkerson, *Flows in Networks*, Princeton, 1962.

[5] T. C. Hu, *Integer Programming and Network Flow*, Addison—Wesley, Reading, Mass., 1970.

[6] A. V. Karzanov and M. V. Lomonosov, Multiflows in Undirected Networks, In: *Mathematical Programming. The Problems of Social and Economic Systems, Operations Research Models* **1** (in Russian) The Institute for Systems Studies, Moscow, 1978.

[7] P. D. Seymour, The Matroids with the Max-flow Min-cut Property. *J. Comb. Theory Ser. B.* **23** (1977) 189—222.

M. V. Lomonosov

Institute for Transmission Problems
Moscow, USSR